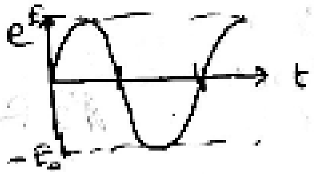
AC Circuits

Though Sir Edison favoured the DC circuits where the current is unidirectional, Tesla's favourite was Alternating current (A.C) which changes direction periodically. For convenient household distributions of electricity using transformers (which can easily step up or down voltages), A.C. has been proven to be much more efficient than D.C. The most used and preferred alternating emf is the sinusoidal one given by

$e = E_0 \sin\omega t$ (or $\cos\omega t$) E_0 is the maximum value and ω is the angular freq.



A complete cycle happens at a time interval, $T = \frac{2\pi}{\omega}$.

* How can we measure this alternating voltage?

If we average it over a complete cycle,

$$\bar{e} = \frac{\int_0^T e\,dt}{T} = 0$$

When averaged over a half cycle, $\bar{e} = \dfrac{\int_0^{T/2} e\,dt}{T/2}$

$$= \frac{2E_0}{T}\int_0^{T/2} \sin\omega t\,dt = \frac{2E_0}{\omega T}\left(-\cos\omega t\right)\Big|_0^{T/2} = 2E_0/\pi$$
$$= 0.637\,E_0$$

But there is no meter which can measure average over a half cycle. So we take the rms value of the voltage, because the Joule heating $\propto$ current^2 and independent of the direction of the current.

$$e_{rms} = \left[\frac{1}{T}\int^T e^2\,dt\right]^{1/2} = \left[\frac{E_0^2}{T}\int_0^T \sin^2\omega t\,dt\right]^{1/2}$$

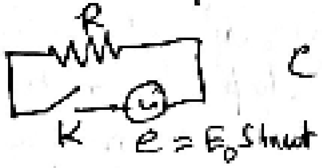
$$= \left[\frac{E_0^2}{2T}\int_0^T (1-\cos 2\omega t)\,dt\right] = \frac{E_0}{\sqrt{2}} \simeq (0.707)E_0$$

$\frac{E_{rms}}{E_{average}}\left(= \frac{0.707\,E_0}{0.637\,E_0} \simeq 1.11\right)$ is known as the 'form factor'.

When we mention or measure the value of AC voltage, we refer to the rms value.

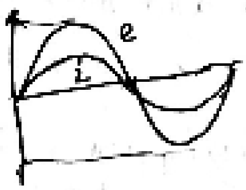
The passive elements of an ac circuit are resistors, inductors and capacitors.

The simplest ac circuit is given by,



Current is given by $i = \frac{e}{R} = \frac{E_0 \sin\omega t}{R}$

$= I_0 \sin\omega t$



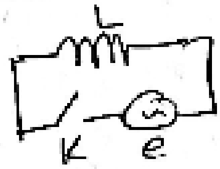
There is no phase difference between e and i. i is said to be 'in phase' with e.

* Power $= e \times i = \frac{E_0^2}{R} \sin^2\omega t$ (instantaneous value)

Average power $\bar{P} = \frac{1}{T}\int_0^T \frac{E_0^2}{R} \sin^2\omega t\, dt = \frac{E_0^2}{2R}$

$\Rightarrow \bar{P} = \frac{E_0}{\sqrt{2}} \frac{E_0}{R\sqrt{2}} = e_{rms}\, i_{rms}$

Pure inductive circuit

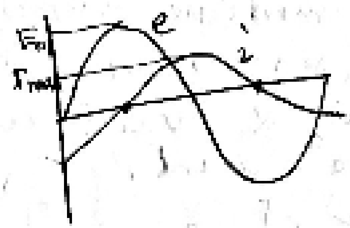


Apply KVL to the circuit,

$e - L\frac{di}{dt} = 0$

$\Rightarrow di = +\frac{e}{L} dt$

$i = +\frac{E_0}{L}\int \sin\omega t\, dt$ (No constant because there is no dc.)



$= -\frac{E_0}{\omega L} \cos\omega t$

$= \frac{E_0}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right) = I_{max} \sin\left(\omega t - \frac{\pi}{2}\right)$

The current is not 'in phase' now, it is lagging the voltage by a phase $\pi/2$.

$\omega L = X_L$ has the dimension of resistance, which is known as inductive reactance, is negligible for very small ω and/or L.

Pure capacitive circuit

Apply KVL $\rightarrow e - \frac{q}{C} = 0$

$q = C E_0 \sin\omega t$

$i = \frac{dq}{dt} = \frac{E_0}{1/\omega C} \cos\omega t = \frac{E_0}{X_C} \sin\left(\omega t + \frac{\pi}{2}\right)$

Current ~~lags~~ leads by a phase $\pi/2$ in capacitive circuit.

$X_c = 1/\omega C$ is known as the capacitive reactance.

X_c is negligible for high ω, C.

* No circuit can be made purely inductive or capacitive because there will always be some resistance causing joule loss of energy.

For a general case, $i = I_0 \sin(\omega t - \phi)$ say,

ϕ is the phase diff. between e and i.

$$\overline{\text{Power}} = \frac{1}{T}\int_0^T e\,i\,dt = \frac{E_0 I_0}{T}\int_0^T \sin\omega t \,\sin(\omega t - \phi)\,dt$$

$$= \frac{E_0 I_0}{T}\int_0^T \sin\omega t\,(\sin\omega t\cos\phi - \cos\omega t\sin\phi)\,dt = \frac{E_0 I_0}{2}\cos\phi$$

$$\boxed{\bar{P} = e_{rms}\, i_{rms} \cos\phi}$$

$\cos\phi$ is known as the power factor.

For pure reactive circuits, $\cos\phi = 0$ and the current is said to be 'wattless'.

RL circuit

KVL gives, $e - L\dfrac{di}{dt} - Ri = 0$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{e}{L} = \frac{E_0}{L}\sin\omega t.$$

Apply operator method for 1st order differential eqn,

$$(R + LD)\,i = e$$

$$i = \frac{e}{R+LD} = \frac{(R-LD)\,e}{(R^2 - L^2D^2)} = \frac{R E_0 \sin\omega t - E_0 \omega L \cos\omega t}{R^2 + \omega^2 L^2}$$

$$= \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\left[\frac{R}{\sqrt{R^2+\omega^2L^2}}\sin\omega t - \frac{\omega L}{\sqrt{R^2+\omega^2L^2}}\cos\omega t\right]$$

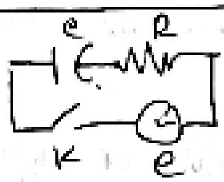
$$\boxed{i = \frac{E_0}{Z}\sin(\omega t - \phi)}$$

where $Z = (R^2 + \omega^2 L^2)^{1/2}$ is known as the IMPEDENCE

& $\tan\phi = \dfrac{\omega L}{R}$

i lags by angle ϕ.

RC circuit



KVL gives $\rightarrow e - \frac{q}{c} - Ri = 0$

$$\frac{1}{c}\frac{dQ}{dt} + R\frac{di}{dt} = \omega E_0 \cos\omega t$$

$$\left(RD + \frac{1}{c}\right) i = \omega E_0 \cos\omega t$$

$$i = \frac{\omega E_0 \cos\omega t}{(RD + 1/c)} = \frac{(RD - 1/c)\,\omega E_0 \cos\omega t}{(R^2D^2 - 1/c^2)}$$

$$= \frac{\omega E_0\left(-R\omega \sin\omega t - \frac{1}{c}\cos\omega t\right)}{-\omega^2R^2 - 1/c^2} = \frac{E_0}{R^2 + \frac{1}{\omega^2c^2}}\left(R\sin\omega t + \frac{1}{\omega c}\cos\omega t\right)$$

$$i = \frac{E_0}{Z}\sin(\omega t + \phi)$$

where $Z = R^2 + 1/\omega^2c^2$ is the impedence

$\phi = \tan^{-1}\left(\frac{1/\omega c}{R}\right)$ is the angle by which i leads e.

These algebraic methods are effective for very simple circuits. But when the circuit becomes complex due to presence of a number of elements, we would require another method called the 'vector method'.

$e = E_0 \sin\omega t = E_0 \,\mathrm{Re}\, e^{j\omega t}$

$e = E_0 \cos\omega t = E_0 \,\mathrm{Im}\, e^{j\omega t}$

From now on, we will use $e = E_0 e^{j\omega t}$ without violating any generality.

For an LR circuit with ac emf $e = E_0 e^{j\omega t}$ we assume that the current is $i = I_0 e^{j\omega t}$ (I_0 is a complex number which will take care of the phase difference, if any).

KVL gives $\rightarrow RI_0 e^{j\omega t} + j\omega L I_0 e^{j\omega t} = E_0 e^{j\omega t}$

$$\Rightarrow I_0 = \frac{E_0}{R+j\omega L}$$

$$i = I_0 e^{j\omega t} = \frac{E_0}{Z} e^{j\omega t} = \frac{E_0}{|Z|} e^{j(\omega t-\phi)} \quad (1)$$

$Z = R + j\omega L$ is the complex impedence.

$$= |Z| e^{j\phi}$$

where $|Z| = \sqrt{R^2+\omega^2 L^2}$ and $\phi = \tan^{-1}\frac{\omega L}{R}$

Similarly for a CR circuit,

$$\frac{q}{C} + Ri = e$$

$$\frac{1}{C} i + R\frac{di}{dt} = j\omega E_0 e^{j\omega t}$$

Assuming $i = I_0 e^{j\omega t}$

$$\frac{1}{C} I_0 e^{j\omega t} + Rj\omega I_0 e^{j\omega t} = j\omega E_0 e^{j\omega t}$$

$$I_0 = \frac{E_0}{R + \frac{1}{j\omega C}} = \frac{E_0}{Z} = \frac{E_0}{|Z|} e^{+j\phi}$$

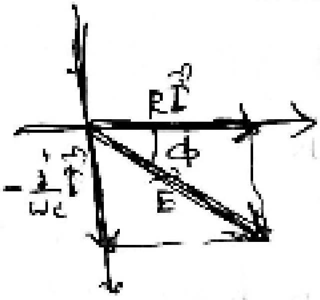
$$\boxed{i = \frac{E_0}{|Z|} e^{j(\omega t+\phi)}} \quad (2)$$ where, $Z = R + \frac{1}{j\omega C}$

$$= R - \frac{j}{\omega C}$$

$$|Z| = \sqrt{R^2 + 1/\omega^2 C^2}$$

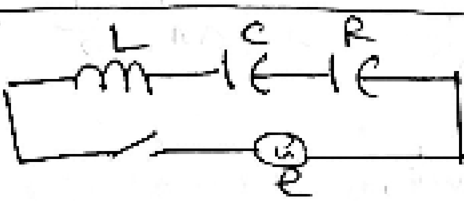
$$\phi = \tan^{-1}\left(\frac{1}{\omega CR}\right)$$

We can express these ~~current~~ voltages in a 'vector diagram'. We take the positive x axis (real axis) as the direction of the voltage across the resistance R (which is in phase with the current). ~~As the~~ As the voltage leads the current in the inductor, we plot $V_L = j\omega L i$ along $+y$ axis of the (positive imaginary axis of the complex plane)



Equation ② for the CR circuit has been expressed in a phasor diagram.

Series LCR circuit



KVL gives

$$L\frac{di}{dt} + \frac{q}{c} + Ri = e$$

$$\Rightarrow L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{c} i = j\omega E_0 e^{j\omega t}$$

Assume $i = I_0 e^{j\omega t}$

$$-\omega^2 L I_0 e^{j\omega t} + j\omega R I_0 e^{j\omega t} + \frac{1}{c} I_0 e^{j\omega t} = j\omega E_0 e^{j\omega t}$$

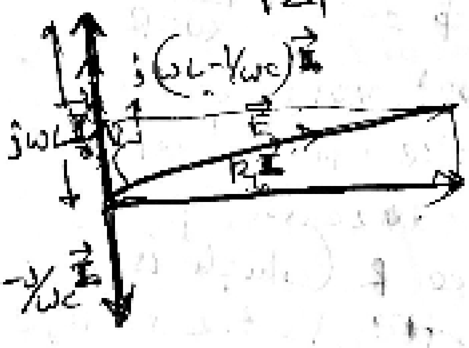
$$I_0 = \frac{E_0}{R + j(\omega L - 1/\omega c)} = \frac{E_0}{Z}$$

$Z = R + j(\omega L - 1/\omega c)$ is the complex impedance

$= |Z| e^{j\phi}$ $\quad |Z| = \left(R^2 + (\omega L - 1/\omega c)^2\right)^{1/2}$

$$\phi = \tan^{-1}\frac{(\omega L - 1/\omega c)}{R}$$

$$i = \frac{E_0}{|Z|} e^{j(\omega t - \phi)} = I_{max} e^{j(\omega t - \phi)}$$

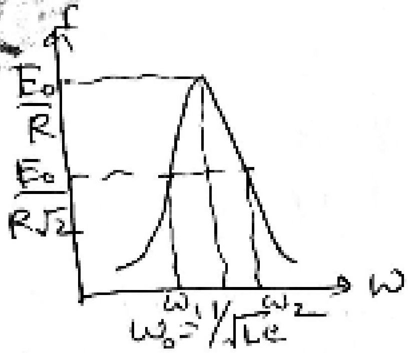


Power factor $= \cos\phi$

$$= \frac{1}{\sqrt{1 + \left(\frac{\omega L - 1/\omega c}{R}\right)^2}}$$

$$= \frac{R}{|Z|}$$

For $\omega L = 1/\omega c$, the current becomes maximum and the circuit becomes purely resistive (e is in phase with i). This is known as RESONANCE



Resonance frequency $\omega_0 = 1/\sqrt{LC}$

The 'sharpness' of the resonance can be measured by measuring the bandwidth $(\Delta\omega)$ of the resonance curve.

$$\Delta\omega = \omega_2 - \omega_1$$

ω_1, ω_2 are the frequency values where the power becomes half of its maximum value.

$$I_{max} = E_0/R$$

$$\frac{I_{max}}{\sqrt{2}} = \frac{E_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \quad \text{(For half power)}$$

Squaring and dividing,

$$\frac{1}{2} = \frac{R^2}{R^2 + (\omega L - 1/\omega C)^2}$$

$$\Rightarrow (\omega L - 1/\omega C)^2 = R^2$$

For $\omega = \omega_1 (< \omega_0)$, (lower half-power-frequency)

$$\frac{1}{\omega_1 C} - \omega_1 L = R \quad (3)$$

similarly, $\omega_2 L - \frac{1}{\omega_2 C} = R \quad (4)$

Adding (3) & (4), we get, $\left(\frac{\omega_2 - \omega_1}{\omega_1 \omega_2}\right)\frac{1}{C} + (\omega_2 - \omega_1)L = 2R$

Subtracting (3) from (4), $\Rightarrow (\omega_2 - \omega_1)L = \frac{\omega_2 - \omega_1}{\omega_1 \omega_2}\left(\frac{1}{C}\right)$

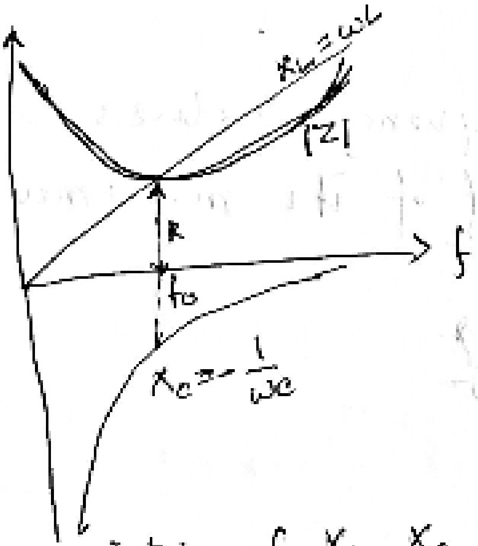
Thus, $(\omega_2 - \omega_1) L = R$

$$\Delta \omega = R/L$$

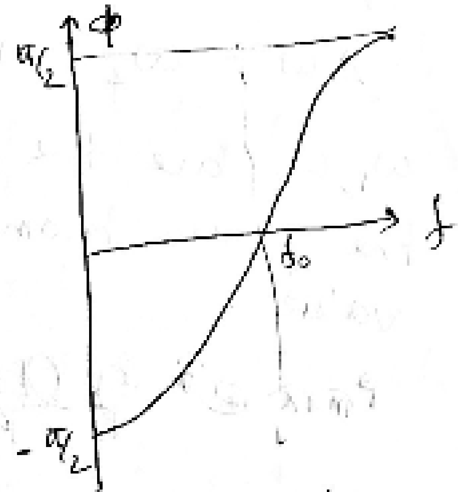
<u>Quality factor</u> (measure of sharpness)

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

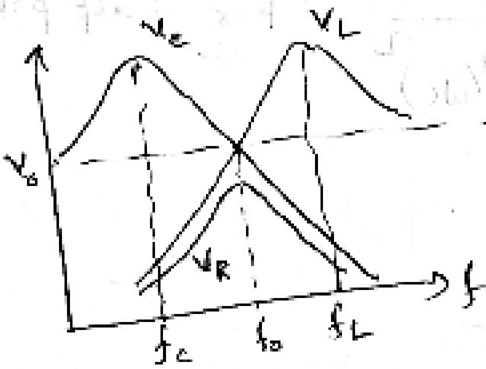
Also, $Q = \frac{\omega_0 L}{R} = \frac{\omega_0 L I}{R I} = \frac{V_L}{V_R}$



variation of X_L, X_C and Z with frequency



Variation of phase angle φ $\tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right)$ with frequency.



variation of V_L, V_C and V_R with frequency.

At $f = 0$ $X_C (1/\omega C) = \infty$, capacitor acts as open, all input voltage V_0 appears across C, and current = 0. As f increases, total reactance $(X_C - X_L)$ decreases causing current to increase. V_C, V_L, V_R increase. At $f = f_0$, V_L and V_C are equal but opposite in phase, V_R reaches its maximum. As f increases further, $(X_L - X_C)$ increases and current decreases, V_C, V_R increases. For $f \to \infty$, V_R and $V_C \to 0$ and $V_L \to V_0$.

As V_R and I are of significant value at and around a very small frequency range, this circuit is known as <u>acceptor circuit</u>.