LAPLACE TRANSFORM

Prepared by

Dr. Raju Dutta
Assistant Professor
Department of Mathematics
Sushil Kar College
Champahati
South 24 Parganas

Laplace Transform and Inverse Laplace Transform

Laplace transform is used in physics and engineering for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. In this analysis, the Laplace transform is often interpreted as a transformation from the *time-domain*, in which inputs and outputs are functions of time, to the *frequency-domain*, where the same inputs and outputs are functions of complex angular frequency, in radians per unit time. Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications. The Laplace transform can also be used for solving ordinary differential equations (intial value problem) and integral equations.

Definition: (Laplace transform)

Let F(t) be a real valued function defined for all $t \geq 0$. The Laplace transform of F(t) is defined to be a function of new variable $f(p)=L[F(t)]=\int_0^\infty e^{-pt} F(t)dt$. The symbol L is called the laplace operator, whose domain is the sets of functions. The variable t in the above integral is of course real, but both F(t) and the variable s may be taken as complex in general case.

We write $F(t)=F_1(t)+iF_2(t)$ and $s=\sigma+i\tau$ and by Euler's formula $e^{i\tau x}=\cos\tau x+i\sin\tau x$, we see that the above integral splits into four integrals of the form $f(p)=L[F(t)]=\int_0^\infty e^{-pt} F(t)dt$ with s replaced by σ and F(t) replaced by the products of $F_1(t)$ or $F_2(t)$ with $\cos\tau x$ or $\sin\tau x$.

Existence of Laplace Transform

Let F(t) is of some exponential order as $t \to \infty$ and is piecewise continuous over every finite interval of $t \geq 0$, then L[F(t)] exists.

[A function F(t) is of exponential order σ as $t \to \infty$ if m(>0) and t_0 can be found such that $|F(t)| < me^{\sigma t}$ for all $t \geq t_0$]

Note: *Until and otherwise stated, once we define a function F(t) defined for all $t \geq 0$, the laplace transform of F(t) exists which is assured by the above result.*

Laplace Transform of some elementary functions

F(t)	L[F(t)] or f(p)
t^n (n is a positive integer)	$\frac{n!}{p^{n+1}}, p > 0$
t^n (n >-1)	$\frac{\Gamma(n+1)}{p^{n+1}}, p > 0$
e^{at}	$\frac{1}{p-a}, p > a$

sin at	$\frac{a}{p^2+a^2}, p>0$
cos at	$\frac{p}{p^2+a^2}, p>0$
sinh at	$\frac{a}{p^2-a^2}, p>\|a\|$
cosh at	$\frac{p}{p^2-a^2}, p>\|a\|$

Some well known properties and results of Laplace transform

1. **Linearity property**

Let c_1 and c_2 be two constants and $F_1(t)$, $F_2(t)$ be two functions defined for all $t\geq0$. Then $L[c_1F_1(t)+c_2F_2(t)]=c_1L[F_1(t)]+c_2L[F_2(t)]$.

2. **First shifting property (Exponential shift)**

Let f(p) be the Laplace transform of F(t), then f(p-a) is the Laplace transform of $e^{at}F(t)$ i.e. $L[e^{at}F(t)]=f(p-a)$.

3. **Second shifting property**

Let L[F(t)]=f(p) and G(t)=F(t-a) if t>a

0 if t<a

Then $L[G(t)]=e^{-ap}f(p)$.

4. **Unit step function or Heaviside's unit function**

The unit step function or Heaviside's unit function H(t-a) is defined as

H(t-a)=1 if t>a

0 if t<a, where a is positive real number.

$L[H(t-a)]=e^{-pt}/p$.

Given any function defined by

$F(t)=F_1(t)$ if $t<a_1$

$F_2(t)$ if $a_1<t<a_2$

$F_3(t)$ if $t>a_2$

Then $F(t)=F_1(t)+\{F_2(t)-F_1(t)\}H(t-a_1)+\{F_3(t)-F_2(t)\}H(t-a_2)$. So, using properties of laplace transform L[F(t)] can easily be determined.

5. **Change of scale property**

   Let L[F(t)]=f(p), then $L[F(at)]=\frac{1}{a}f(p/a)$, where 'a' is a non zero constant.

6. **Laplace transform of a periodic function**

   Let F(t) be a periodic function of period 'a' and F(t) is piecewise continuous in $0<t<a$,

then $L[F(t)]=\frac{1}{1-e^{-pa}}\int_0^a e^{-pt}F(t)dt$.

7. **Laplace transform of derivatives of nth order**

   Let F(t) and its derivative $F'(t), F''(t),\ldots,F^{n-1}(t)$ be continuous for all $t\geq 0$ and is of exponential order σ as $t\to\infty$, then the Laplace transform of $F^n(t)$ exists when $p>\sigma$ and is given by

$$L[F^n(t)]=p^nL[F(t)]-p^{n-1}F(0)-p^{n-2}F'(0)-\ldots\ldots.-F^{n-1}(0).$$

In particular, $L[F'(t)]=pL[F(t)]-F(0)$

$$L[F''(t)]=p^2L[F(t)]-pF'(0)-F(0).$$

8. (a) Let L[F(t)]=f(p), then $L[F(t)/t]=\int_p^\infty f(p)dp$, provided the integral exists.

   (b) Let L[F(t)]=f(p), then $L[\int_0^t F(x)dx]=\frac{f(p)}{p}$.

Definition: (Inverse Laplace transform)

If L[F(t)]=f(p), then F(t) is called the inverse laplace transform of f(p) and we write $F(t)=L^{-1}[f(p)]$. The symbol L^{-1} is called the inverse Laplace operator.

Formulae of Inverse Laplace transform

$f(p)$	$L^{-1}[f(p)]$ or $F(t)$
$\frac{1}{p^{n+1}}$, n is a positive integer	$t^n/n!$
$\frac{1}{p^{n+1}}$, $n > -1$	$\frac{t^n}{\Gamma(n+1)}$
$\frac{1}{p-a}$	e^{at}
$\frac{1}{p^2+a^2}$	$(\sin at)/a$
$\frac{p}{p^2+a^2}$	$\cos at$
$\frac{1}{p^2-a^2}$	$(\sinh at)/a$
$\frac{p}{p^2-a^2}$	$\cosh at$

Some well-known properties and results of Inverse Laplace transform

1. **Linearity property**

Let c_1 and c_2 be two constants and $F_1(t)$, $F_2(t)$ be two functions defined for all $t \geq 0$ such that $L[F_1(t)]=f_1(p)$ and $L[F_2(t)]=f_2(p)$.

Then $L^{-1}[c_1 f_1(p) + c_2 f_2(p)] = c_1 L^{-1}[f_1(p)] + c_2 L^{-1}[f_2(p)]$.

2. **First shifting property (Exponential shift)**

Let $f(p)$ be the Laplace transform of $F(t)$, then $L^{-1}[f(p-a)] = e^{at} L^{-1}[f(p)]$.

3. **Second shifting property**

Let $L^{-1}[f(p)]=F(t)$, then $L^{-1}[e^{ap}f(p)]=G(t)=F(t-a)$ if $t>a$

0 if $t<a$

4. **Change of scale property**

Let $L^{-1}[f(p)]=F(t)$, then $L^{-1}[f(ap)]=\frac{1}{a}F(t/a)$, where 'a' is a non zero constant.

5. If $L^{-1}[f(p)]=F(t)$, then $L^{-1}[f^n(p)]=L^{-1}\left[\frac{d^n(p)}{dp^n}\right]=(-1)^n t^n F(t)$, $n=1,2,3,\ldots.$

Convolution:

Let F(t) and G(t) be two continuous functions for all $t\geq0$, then the convolution of F(t) and G(t), denoted by F*G, is defined by

$$F*G=\int_0^t F(u)G(t-u)du$$

Properties of convolution

1. $F*G=G*F$, 2. $F*(G+H)=F*G+F*H$.

Convolution Theorem:

Let F(t) and G(t) be two rela valued function defined for all $t\geq0$.

If $L^{-1}[f(p)]=F(t)$ and $L^{-1}[g(p)]=G(t)$, then $L^{-1}[f(p)g(p)]=\int_0^t F(u)G(t-u)du$.

The Laplace Transform

Definition and properties of Laplace Transform, piecewise continuous functions, the Laplace Transform method of solving initial value problems

The method of Laplace transforms is a system that relies on algebra (rather than calculus-based methods) to solve linear differential equations. While it might seem to be a somewhat cumbersome method at times, it is a very powerful tool that enables us to readily deal with linear differential equations with discontinuous forcing functions.

Definition: Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$, denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$, is an *integral transform* given by the *Laplace integral*:

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt}$$

Provided that this (improper) integral exists, i.e. that the integral is convergent.

The Laplace transform is an operation that transforms a function of t (i.e., a function of *time domain*), defined on $[0, \infty)$, to a function of s (i.e., of *frequency domain*)[*]. $F(s)$ is the *Laplace transform*, or simply *transform*, of $f(t)$. Together the two functions $f(t)$ and $F(s)$ are called a *Laplace transform pair*.

For functions of t continuous on $[0, \infty)$, the above transformation to the frequency domain is *one-to-one*. That is, different continuous functions will have different transforms.

---

[*] The *kernel* of the Laplace transform, e^{-st} in the integrand, is unit-less. Therefore, the unit of s is the reciprocal of that of t. Hence s is a variable denoting (complex) frequency.

Example: Let $f(t) = 1$, then $F(s) = \frac{1}{s}$, $s > 0$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st}\, dt = \frac{-1}{s} e^{-st} \Big|_0^\infty$$

The integral is divergent whenever $s \le 0$. However, when $s > 0$, it converges to

$$\frac{-1}{s}\left(0 - e^0\right) = \frac{-1}{s}(-1) = \frac{1}{s} = F(s).$$

Example: Let $f(t) = t$, then $F(s) = \frac{1}{s^2}$, $s > 0$.

[This is left to you as an exercise.]

Example: Let $f(t) = e^{at}$, then $F(s) = \frac{1}{s-a}$, $s > a$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$$

The integral is divergent whenever $s \le a$. However, when $s > a$, it converges to

$$\frac{1}{a-s}\left(0 - e^0\right) = \frac{1}{a-s}(-1) = \frac{1}{s-a} = F(s).$$

Definition: A function $f(t)$ is called *piecewise continuous* if it only has finitely many (or none whatsoever – a continuous function is considered to be "piecewise continuous"!) discontinuities on any interval $[a, b]$, and that both one-sided limits exist as t approaches each of those discontinuity from <u>within</u> the interval. The last part of the definition means that f could have removable and/or jump discontinuities only; it cannot have any infinity discontinuity.

Theorem: Suppose that

1. f is piecewise continuous on the interval $0 \le t \le A$ for any $A > 0$.
2. $|f(t)| \le Ke^{at}$ when $t \ge M$, for any real constant a, and some positive constants K and M. (This means that f is "of *exponential order*", i.e. its rate of growth is no faster than that of exponential functions.)

Then the Laplace transform, $F(s) = \mathcal{L}\{f(t)\}$, exists for $s > a$.

Note: The above theorem gives a <u>sufficient</u> condition for the existence of Laplace transforms. It is <u>not</u> a <u>necessary</u> condition. A function does not need to satisfy the two conditions in order to have a Laplace transform. Examples of such functions that nevertheless have Laplace transforms are logarithmic functions and the unit impulse function.

Some properties of the Laplace Transform

1. $\mathcal{L}\{0\} = 0$

2. $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$

3. $\mathcal{L}\{cf(t)\} = c\,\mathcal{L}\{f(t)\}$, for any constant c.

Properties 2 and 3 together means that the Laplace transform is *linear*.

4. **[The derivative of Laplace transforms]**

$\mathcal{L}\{(-t)f(t)\} = F'(s)$ or, equivalently $\mathcal{L}\{t\,f(t)\} = -F'(s)$

Example: $\mathcal{L}\{t^2\} = -(\mathcal{L}\{t\})' = -\frac{d}{ds}\frac{1}{s^2} = -\frac{-2}{s^3} = \frac{2}{s^3}$

In general, the derivatives of Laplace transforms satisfy

$\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s)$ or, equivalently $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$

Warning: The Laplace transform, while a linear operation, is **not** *multiplicative*. That is, in general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.$$

Exercise: (a) Use property 4 above, and the fact that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, to deduce that $\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$. (b) What will $\mathcal{L}\{t^2 e^{at}\}$ be?

Exercises C-1.1:

1 – 5 Use the (integral transformation) definition of the Laplace transform to find the Laplace transform of each function below.

1. t^2
2. te^{6t}
3. $\cos 3t$
4. $e^{-t}\sin 2t$
5.* e^{iat}, where i and α are constants, $i = \sqrt{-1}$.

6 – 8 Each function $F(s)$ below is defined by a definite integral. Without integrating, find an explicit expression for each $F(s)$.
[Hint: each expression is the Laplace transform of a certain function. Use your knowledge of Laplace Transformation, or with the help of a table of common Laplace transforms to find the answer.]

6. $\int_0^\infty e^{-(s+7)t}\,dt$
7. $\int_0^\infty t^2 e^{-(s-3)t}\,dt$
8. $\int_0^\infty 4e^{-st}\sin 6t\,dt$

Answers C-1.1:

1. $\dfrac{2}{s^3}$
2. $\dfrac{1}{(s-6)^2}$
3. $\dfrac{s}{s^2+9}$
4. $\dfrac{2}{s^2+2s+5}$
5. $\dfrac{s}{s^2+\alpha^2} + i\dfrac{\alpha}{s^2+\alpha^2}$

Note: Since the Euler's formula says that $e^{iat} = \cos at + i\sin at$, therefore, $\mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at + i\sin at\}$. That is, the real part of its Laplace transform corresponds to that of $\cos at$, the imaginary part corresponds to that of $\sin at$. (Check it for yourself!)

6. $\dfrac{1}{s+7}$
7. $\dfrac{2}{(s-3)^3}$
8. $\dfrac{24}{s^2+36}$

Solution of Initial Value Problems

We now shall meet "the new System": how the Laplace transforms can be used to solve linear differential equations algebraically.

Theorem: [**Laplace transform of derivatives**] Suppose f is of exponential order, and that f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$. Then

$$\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$$

Applying the theorem multiple times yields:

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - sf(0) - f'(0),$$

$$\mathcal{L}\{f'''(t)\} = s^3\,\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0),$$

$$\vdots$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots$$
$$- s^2 f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

This is an extremely useful aspect of the Laplace transform: that it changes differentiation with respect to t into multiplication by s (and, as seen a little earlier, differentiation with respect to s into multiplication by $-t$, on the other hand). Equally importantly, it says that the Laplace transform, when applied to a differential equation, would change derivatives into algebraic expressions in terms of s and (the transform of) the dependent variable itself. Thus, it can transform a differential equation into an algebraic equation.

We are now ready to see how the Laplace transform can be used to solve differentiation equations.

Solving initial value problems using the method of Laplace transforms

To solve a linear differential equation using Laplace transforms, there are only 3 basic steps:

1. Take the Laplace transforms of both sides of an equation.

2. Simplify algebraically the result to solve for $\mathscr{L}\{y\} = Y(s)$ in terms of s.

3. Find the inverse transform of $Y(s)$. (Or, rather, find a function $y(t)$ whose Laplace transform matches the expression of $Y(s)$.) This inverse transform, $y(t)$, is the solution of the given differential equation.

The nice thing is that the same 3-step procedure works whether or not the differential equation is homogeneous or nonhomogeneous. The first two steps in the procedure are rather mechanical. The last step is the heart of the process, and it will take some practice. Let's get started.

Example: $y'' - 6y' + 5y = 0,$ $y(0) = 1,\ \ y'(0) = -3$

[Step 1] Transform both sides

$$\mathcal{L}\{y'' - 6y' + 5y\} = \mathcal{L}\{0\}$$

$$(s^2\mathcal{L}\{y\} - sy(0) - y'(0)) - 6(s\mathcal{L}\{y\} - y(0)) + 5\mathcal{L}\{y\} = 0$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s^2\mathcal{L}\{y\} - s - (-3)) - 6(s\,\mathcal{L}\{y\} - 1) + 5\mathcal{L}\{y\} = 0$$

$$(s^2 - 6s + 5)\,\mathcal{L}\{y\} - s + 9 = 0$$

$$(s^2 - 6s + 5)\,\mathcal{L}\{y\} = s - 9$$

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5}$$

[Step 3] Find the inverse transform $y(t)$.

Use partial fractions to simplify,

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{a}{s-1} + \frac{b}{s-5}$$

$$\frac{s-9}{s^2 - 6s + 5} = \frac{a(s-5)}{(s-1)(s-5)} + \frac{b(s-1)}{(s-5)(s-1)}$$

$$s - 9 = a(s-5) + b(s-1) = (a+b)s + (-5a - b)$$

Equating the corresponding coefficients:

$$\begin{aligned} 1 &= a + b \\ -9 &= -5a - b \end{aligned} \qquad\qquad \begin{aligned} a &= 2 \\ b &= -1 \end{aligned}$$

Hence,

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{2}{s-1} - \frac{1}{s-5}.$$

The last expression corresponds to the Laplace transform of $2e^{t} - e^{5t}$. Therefore, it must be that

$$y(t) = 2e^{t} - e^{5t}.$$

Many of the observant students no doubt have noticed an interesting aspect (out of many) of the method of Laplace transform: that it finds the particular solution of an initial value problem directly, without solving for the general solution first. Indeed, it usually takes more effort to find the general solution of an equation than it takes to find a particular solution!

The Laplace Transform method can be used to solve linear differential equations of any order, rather than just second order equations as in the previous example. The method will also solve a nonhomogeneous linear differential equation directly, using the exact same three basic steps, without having to separately solve for the complementary and particular solutions. These points are illustrated in the next two examples.

Example: $y' + 2y = 4te^{-2t}, \qquad y(0) = -3.$

[Step 1] Transform both sides

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{4te^{-2t}\}$$

$$(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{4te^{-2t}\} = \frac{4}{(s+2)^2}$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s\mathcal{L}\{y\} - (-3)) + 2\mathcal{L}\{y\} = \frac{4}{(s+2)^2}$$

$$(s+2)\,\mathcal{L}\{y\} + 3 = \frac{4}{(s+2)^2}$$

$$(s+2)\,\mathcal{L}\{y\} = \frac{4}{(s+2)^2} - 3$$

$$\mathcal{L}\{y\} = \frac{4}{(s+2)^3} - \frac{3}{s+2} = \frac{4 - 3(s+2)^2}{(s+2)^3} = \frac{-3s^2 - 12s - 8}{(s+2)^3}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathcal{L}\{y\} = \frac{-3s^2 - 12s - 8}{(s+2)^3} = \frac{a}{(s+2)^3} + \frac{b}{(s+2)^2} + \frac{c}{s+2}.$$

$$\frac{-3s^2-12s-8}{(s+2)^3}=\frac{a}{(s+2)^3}+\frac{b(s+2)}{(s+2)^3}+\frac{c(s+2)^2}{(s+2)^3}$$

$$=\frac{a+bs+2b+cs^2+4cs+4c}{(s+2)^3}=\frac{cs^2+(b+4c)s+(a+2b+4c)}{(s+2)^3}$$

$$\begin{aligned}-3&=c\\-12&=b+4c\\-8&=a+2b+4c\end{aligned}\qquad\qquad\begin{aligned}a&=4\\b&=0\\c&=-3\end{aligned}$$

$$\mathcal{L}\{y\}=\frac{-3s^2-12s-8}{(s+2)^3}=\frac{4}{(s+2)^3}-\frac{3}{s+2}.$$

This expression corresponds to the Laplace transform of $2t^2e^{-2t}-3e^{-2t}$. Therefore,

$$y(t)=2t^2e^{-2t}-3e^{-2t}.$$

Note: $\mathcal{L}\{t^n e^{at}\}=\dfrac{n!}{(s-a)^{n+1}}$

Example: $y'' - 3y' + 2y = e^{3t}, \qquad y(0) = 1, \quad y'(0) = 0$

[Step 1] Transform both sides

$$(s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0)) - 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{3t}\}$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s^2 \mathcal{L}\{y\} - s - 0) - 3(s\mathcal{L}\{y\} - 1) + 2\mathcal{L}\{y\} = 1/(s-3)$$

$$(s^2 - 3s + 2)\,\mathcal{L}\{y\} - s + 3 = 1/(s-3)$$

$$(s^2 - 3s + 2)\,\mathcal{L}\{y\} = s - 3 + \frac{1}{s-3} = \frac{(s-3)^2 + 1}{s-3}$$

$$\mathcal{L}\{y\} = \frac{s^2 - 6s + 10}{(s^2 - 3s + 2)(s-3)} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$$

[Step 3] Find the inverse transform $y(t)$.

By partial fractions,

$$\mathcal{L}\{y\} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{5}{2}\frac{1}{s-1} - 2\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}.$$

Therefore, $y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$.

For the next example, we will need the following Laplace transforms:

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}, \qquad s>0$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \qquad s>0$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}, \qquad s>a$$

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}, \qquad s>a$$

Note: The values of a and b in the last two expressions' denominators can be determined without using the method of completing the squares. Any irreducible quadratic polynomial $s^2 + Bs + C$ can always be written in the required from of $(s-a)^2 + b^2$ by using the quadratic formula to find (necessarily complex-valued roots) s. The value a is the real part of s, and the value b is just the absolute value of the imaginary part of s. That is, if $s = \lambda \pm \mu i$, then $a = \lambda$ and $b = \mu$.

Example: $y'' - 2y' + 2y = \cos(t), \qquad y(0) = 1, \; y'(0) = 0$

[Step 1] Transform both sides

$$(s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0)) - 2(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos(t)\}$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s^2 \mathcal{L}\{y\} - s - 0) - 2(s\mathcal{L}\{y\} - 1) + 2\mathcal{L}\{y\} = s / (s^2 + 1)$$

$$(s^2 - 2s + 2)\,\mathcal{L}\{y\} - s + 2 = s/(s^2 + 1)$$

$$(s^2 - 2s + 2)\,\mathcal{L}\{y\} = s - 2 + \frac{s}{s^2+1} = \frac{(s-2)(s^2+1)+s}{s^2+1}$$

$$\mathcal{L}\{y\} = \frac{s^3 - 2s^2 + s - 2 + s}{(s^2+1)(s^2-2s+2)} = \frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathcal{L}\{y\} = \frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} + \frac{4s-6}{s^2-2s+2}\right]$$

$$= \frac{1}{5}\left[\frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{4(s-1)}{s^2-2s+2} - \frac{2}{s^2-2s+2}\right]$$

which corresponds to

$$y(t) = \frac{1}{5}\left[\cos(t) - 2\sin(t) + 4e^{t}\cos(t) - 2e^{t}\sin(t)\right]$$

Examples: Find the inverse Laplace transform of each

(i) $F(s) = \dfrac{2s-5}{s^2+4s+8}$

Rewrite $F(s)$ as:

$$F(s) = \frac{2s-5}{(s+2)^2+2^2} = \frac{2(s+2)}{(s+2)^2+2^2} - \frac{9}{2}\frac{2}{(s+2)^2+2^2}$$

Answer: $f(t) = 2e^{-2t}\cos(2t) - \dfrac{9}{2}e^{-2t}\sin(2t)$

(ii) $F(s) = \dfrac{s+4}{(s-2)^3}$

Use partial fractions to rewrite $F(s)$ as:

$$F(s) = \frac{1}{(s-2)^2} + \frac{6}{(s-2)^3}$$

Answer: $f(t) = te^{2t} + 3t^2e^{2t}$

Exercises C-1.2:

1 – 9 Find the Laplace transform of each function below.

1. $f(t) = t^3 - t^2 + 5t + 2$

2. $f(t) = -2\cos 6t + 5\sin 6t$

3. $f(t) = 3e^t - 4e^{2t} + 2e^{-4t}$

4. $f(t) = 7t + 6e^t - 2e^{-t} - 10$

5. (a) $f(t) = \sin 2t \sin 3t$ (b) $f(t) = \sin 2t \cos 2t$

6. (a) $f(t) = \cos^2 5t$ (b) $f(t) = t\cos^2 5t$

7. (a) $f(t) = te^{at}\cos bt,$ (b) $f(t) = te^{at}\sin bt$

8. $f(t) = t^3 \cos 2t$

9. (a) $f(t) = \cos(\alpha t + \beta)$ (b) $f(t) = \sin(\alpha t + \beta)$

10 – 18 Find the inverse Laplace transform of each function below.

10. $F(s) = \dfrac{4s+2}{s^2+6s+34}$ 11. $F(s) = \dfrac{1}{(s-2)(s-4)(s-8)}$

12. $F(s) = \dfrac{3s-7}{4s^2+1}$ 13. $F(s) = \dfrac{s^2+s-6}{s^3+2s^2+s}$

14. $F(s) = \dfrac{1}{s^4-81}$ 15. $F(s) = \dfrac{s^3}{s^4-16}$

16. $F(s) = \dfrac{10}{s^4-s^3}$ 17. $F(s) = \dfrac{12}{s^3-8}$

18. $F(s) = \dfrac{1}{(s-\alpha)(s-\beta)}$

19 – 32 Use the method of Laplace transforms to solve each IVP.

19. $y' + 10y = t^2$, $y(0) = 0$

20. $y' + 2y = te^{-t}$, $y(0) = 2$

21. $y' - 6y = 2\sin 3t$, $y(0) = -1$

22. $y'' + 2y' = te^{-t}$, $y(0) = 6$, $y'(0) = -1$

23. $y'' + 4y' - 5y = 0$, $y(0) = 5$, $y'(0) = -1$

24. $y'' - 2y' + y = 2t - 3$, $y(0) = 5$, $y'(0) = 11$

25. $y'' + 8y' + 25y = 13e^{-2t}$, $y(0) = -1$, $y'(0) = 18$

26. $y'' + 6y' + 34y = 0$, $y(0) = -1$, $y'(0) = 13$

27. $y'' + 4y = 8\cos 2t - 8e^{-2t}$, $y(0) = -2$, $y'(0) = 0$

28. $y'' - 4y' + 4y = 8t^2 - 16t + 4$, $y(0) = 0$, $y'(0) = 1$

29. $y'' - 4y' - 5y = 3t^3$, $y(0) = 3$, $y'(0) = 3$

30. $y''' + 3y'' + 3y' + y = 0$, $y(0) = 7$, $y'(0) = -7$, $y''(0) = 11$

31. $y''' + 4y'' - 5y' = 0$, $y(0) = 4$, $y'(0) = -7$, $y''(0) = 23$

32. $y''' - y'' + 4y' - 4y = 26e^{3t}$, $y(0) = -2$, $y'(0) = 3$, $y''(0) = 1$

33. Prove the time-scaling property (property III, Appendix A).

Hint: Let $u = ct$, and $v = s/c$, then show $\mathcal{L}\{f(ct)\} = \frac{1}{c}\int_0^\infty e^{-vu} f(u)\,du$.

34. Use property IV of Appendix A to verify that

$$\mathcal{L}^{-1}\{\arctan(a/s)\} = \frac{1}{t}\sin at.$$

Answers C-1.2:

1. $F(s)=\dfrac{2s^3+5s^2-2s+6}{s^4}$ 2. $F(s)=\dfrac{-2s+30}{s^2+36}$

3. $F(s)=\dfrac{s^2-12s-4}{(s-1)(s-2)(s+4)}$ 4. $F(s)=\dfrac{-6s^3+15s^2+10s-7}{s^4-s^2}$

5. (a) $F(s)=\dfrac{s}{2(s^2+1)}-\dfrac{s}{2(s^2+25)}$ (b) $F(s)=\dfrac{2}{s^2+16}$

6. (a) $F(s)=\dfrac{1}{2}\left[\dfrac{1}{s}+\dfrac{s}{s^2+100}\right]$, (b) $F(s)=\dfrac{1}{2}\left[\dfrac{1}{s^2}+\dfrac{s^2-100}{(s^2+100)^2}\right]$

7. (a) $F(s)=\dfrac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$, (b) $F(s)=\dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$

8. $F(s)=\dfrac{6s^4-144s^2+96}{(s^2+4)^4}$

9. (a) $F(s)=\dfrac{s\cos(\beta)-\alpha\sin(\beta)}{s^2+\alpha^2}$, (b) $F(s)=\dfrac{s\sin(\beta)+\alpha\cos(\beta)}{s^2+\alpha^2}$

10. $f(t)=4e^{-3t}\cos 5t-2e^{-3t}\sin 5t$

11. $f(t)=\dfrac{1}{12}e^{2t}-\dfrac{1}{8}e^{4t}+\dfrac{1}{24}e^{8t}$

12. $f(t)=\dfrac{3}{4}\cos\dfrac{t}{2}-\dfrac{7}{2}\sin\dfrac{t}{2}$

13. $f(t)=7e^{-t}+6te^{-t}-6$

14. $f(t)=\dfrac{1}{108}e^{3t}-\dfrac{1}{108}e^{-3t}-\dfrac{1}{54}\sin 3t$

15. $f(t)=\dfrac{1}{4}e^{2t}+\dfrac{1}{4}e^{-2t}-\dfrac{1}{2}\cos 2t$

16. $f(t)=10e^{t}-5t^2-10t-10$

17. $f(t)=e^{2t}-e^{-t}\cos(\sqrt{3}\,t)-\sqrt{3}\,e^{-t}\sin(\sqrt{3}\,t)$

18. $f(t)=\dfrac{1}{\alpha-\beta}(e^{\alpha t}-e^{\beta t})$

19. $y=\dfrac{t^2}{10}-\dfrac{t}{50}+\dfrac{1}{500}-\dfrac{1}{500}e^{-10t}$

20. $y=te^{-t}-e^{-t}+3e^{-2t}$

21. $y = \frac{-13}{15}e^{6t} - \frac{2}{15}\cos 3t - \frac{4}{15}\sin 3t$
22. $y = -te^{-t} + 6$
23. $y = 4e^{t} + e^{-5t}$
24. $y = 4e^{t} + 5te^{t} + 2t + 1$
25. $y = -2e^{-4t}\cos 3t + 4e^{-4t}\sin 3t + e^{-2t}$
26. $y = -e^{-3t}\cos 5t + 2e^{-3t}\sin 5t$
27. $y = -\cos 2t - \sin 2t + 2t\sin 2t - e^{-2t}$
28. $y = te^{2t} + 2t^{2}$
29. $y = e^{5t} + 2e^{-t} + 3t^{3}$
30. $y = 7e^{-t} + 2t^{2}e^{-t}$
31. $y = 5 - 2e^{t} + e^{-5t}$
32. $y = -4e^{t} + e^{3t} + \cos 2t + 2\sin 2t$

The Inverse Laplace Transform

1. If $\mathcal{L}\{f(t)\} = F(s)$, then the *inverse Laplace transform* of $F(s)$ is

$$\mathcal{L}^{-1}\{F(s)\} = f(t). \tag{1}$$

The inverse transform $\mathcal{L}^{-1}$ is a linear operator:

$$\mathcal{L}^{-1}\{F(s) + G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\}, \tag{2}$$

and

$$\mathcal{L}^{-1}\{cF(s)\} = c\mathcal{L}^{-1}\{F(s)\}, \tag{3}$$

for any constant c.

2. **Example:** The inverse Laplace transform of

$$U(s) = \frac{1}{s^3} + \frac{6}{s^2+4},$$

is

$$\begin{aligned} u(t) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \frac{s^2}{2} + 3\sin 2t. \end{aligned} \tag{4}$$

3. **Example:** Suppose you want to find the inverse Laplace transform $x(t)$ of

$$X(s) = \frac{1}{(s+1)^4} + \frac{s-3}{(s-3)^2+6}.$$

Just use the shift property (paragraph 11 from the previous set of notes):

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+6}\right\} \\ &= \frac{e^{-t}\,t^3}{6} + e^{3t}\cos\sqrt{6}t. \end{aligned}$$

4. **Example:** Let $y(t)$ be the inverse Laplace transform of

$$Y(s) = \frac{e^{-3s}\,s}{s^2+4}.$$

Don't worry about the exponential term. Since the inverse transform of $s/(s^2+4)$ is $\cos 2t$, we have by the switchig property (paragraph 12 from the previous notes):

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-3s}\,s}{s^2+4}\right\} \\ &= H(t-3)\cos 2(t-3). \end{aligned}$$

5. **Example:** Let $G(s) = s(s^2+4s+5)^{-1}$. The inverse transform of $G(s)$ is

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+1}\right\} \\ &= e^{-2t}\cos t - 2e^{-2t}\sin t. \end{aligned} \tag{5}$$

6. There is usually more than one way to invert the Laplace transform. For example, let $F(s) = (s^2+4s)^{-1}$. You could compute the inverse transform of this function by completing the square:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2-4}\right\} \\ &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2-4}\right\} \\ &= \frac{1}{2}\,e^{-2t}\sinh 2t. \end{aligned} \tag{6}$$

You could also use the partial fraction decomposition (PFD) of $F(s)$:

$$F(s) = \frac{1}{s(s+4)} = \frac{1}{4s} - \frac{1}{4(s+4)}.$$

Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{4s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{4(s+4)}\right\} \\ &= \frac{1}{4} - \frac{1}{4}e^{-4t} \\ &= \frac{1}{2}\,e^{-2t}\sinh 2t. \end{aligned} \tag{7}$$

7. Example: Compute the inverse Laplace transform $q(t)$ of

$$Q(s) = \frac{3s}{(s^2+1)^2}.$$

You could compute $q(t)$ by partial fractions, but there's a less tedious way. Note that

$$Q(s) = -\frac{3}{2}\frac{d}{ds}\frac{1}{s^2+1}.$$

Hence,

$$\begin{aligned} q(t) &= \mathcal{L}^{-1}\left\{Q(s)\right\} \\ &= -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{d}{ds}\frac{1}{s^2+1}\right\} \\ &= \frac{3}{2}t\sin t. \end{aligned} \tag{8}$$

8. Definition: The *convolution* of functions $f(t)$ and $g(t)$ is

$$(f*g)(t) = \int_0^t f(t-v)g(v)\,dv. \tag{9}$$

As we showed in class, the convolution is commutative:

$$(f*g)(t) = \int_0^t f(t-v)g(v)\,dv = \int_0^t g(t-v)f(v)\,dv = (g*f)(t). \tag{10}$$

9. Example: Let $f(t) = t$ and $g(t) = e^t$. The convolution of f and g is

$$\begin{aligned} (f*g)(t) &= \int_0^t (t-v)e^v\,dv \\ &= t\int_0^t e^v\,dv - \int_0^t ve^v\,dv \\ &= e^t - t - 1. \end{aligned} \tag{11}$$

10. Proposition: (The Convolution Theorem) If the Laplace transforms of $f(t)$ and $g(t)$ are $F(s)$ and $G(s)$ respectively, then

$$\mathcal{L}\left\{(f*g)(t)\right\} = F(s)G(s), \tag{12}$$

that is,

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = (f * g)(t). \tag{13}$$

11. Suppose that you want to find the inverse transform $x(t)$ of $X(s)$. If you can write $X(s)$ as a product $F(s)G(s)$ where $f(t)$ and $g(t)$ are known, then by the above result, $x(t) = (f * g)(t)$.

12. **Example:** Consider the previous example: Find the inverse transform $q(s)$ of

$$Q(s) = \frac{3s}{(s^2+1)^2}.$$

Write $Q(s) = F(s)G(s)$, where

$$F(s) = \frac{3}{s^2+1},$$

and

$$G(s) = \frac{s}{s^2+1}.$$

The inverse transforms are of $F(s)$ and $G(s)$ are $f(t) = 3\sin t$ and $g(t) = \cos t$. Therefore

$$\begin{aligned} q(s) &= \mathcal{L}^{-1}\left\{Q(s)\right\} \\ &= \mathcal{L}^{-1}\left\{F(s)G(s)\right\} \\ &= (f * g)(t) \\ &= 3\int_0^t \sin(t-v)\cos v\,dv. \end{aligned} \tag{14}$$

Even if you stop here, you at least have a fairly simple, compact expression for $q(s)$. To do the integral (14), use the trigonometric identity

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}.$$

With this, (14) becomes

$$\begin{aligned} q(s) &= \frac{3}{2}\int_0^t \sin t\,dv + \int_0^t \sin(t-2v)\,dv \\ &= \frac{3}{2}t\sin t. \end{aligned} \tag{15}$$

13. **Example:** Find the inverse Laplace transform $x(t)$ of the function

$$X(s) = \frac{1}{s(s^2+4)}.$$

If you want to use the convolution theorem, write $X(s)$ as a product:

$$X(s) = \frac{1}{s} \frac{1}{s^2 + 4}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2} \sin 2t,$$

we have

$$\begin{aligned} x(t) &= \frac{1}{2} \int_0^t \sin 2v \, dv \\ &= \frac{1}{4}(1 - \cos 2t). \end{aligned}$$

You could also use the PFD:

$$X(s) = \frac{1}{4s} - \frac{s}{4(s^2 + 4)}.$$

Therefore,

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{4s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{4(s^2 + 4)}\right\} \\ &= \frac{1}{4}(1 - \cos 2t). \end{aligned}$$

Quick way to compute the inverse Laplace transform (without partial fractions).

1. Rational functions with only simple roots in the denominator.

In this case the solutions can be simply written down. It is best to illustrate by examples.

Example 1. Find the Laplace inverse of

$$(\mathcal{L}f)(s) = \frac{1}{(s-1)(s-2)}$$

The Laplace inverse is given by multiplying the above function by e^{st} and then estimating the resulting function at $s=1$, with the factor $s-1$ taken out, plus, the resulting function estimated at $s=2$, with the factor $s-2$ taken out. Namely

$$f(t) = \frac{e^{st}}{s-1}|_{s=2} + \frac{e^{st}}{s-2}|_{s=1} = \frac{e^{2t}}{2-1} + \frac{e^{t}}{1-2} = e^{2t} - e^{t}$$

The method still works if we have more factors in the bottom, or if the top is multiplied by a constant (or in fact any polynomial of degree less than the bottom).

Example 2. Find the inverse Laplace transform of

$$(\mathcal{L}f)(s) = \frac{2}{(s-1)(s-2)(s-3)}$$

Solution: We simply *write it down*,

$$\begin{aligned} f(t) &= \frac{2\,e^{st}}{(s-2)(s-3)}|_{s=1} + \frac{2\,e^{st}}{(s-1)(s-3)}|_{s=2} + \frac{2\,e^{st}}{(s-1)(s-2)}|_{s=3} \\ &= \frac{2\,e^{t}}{(1-2)(1-3)} + \frac{2\,e^{2t}}{(2-1)(2-3)} + \frac{2\,e^{3t}}{(3-1)(3-2)} \\ &= \frac{2\,e^{t}}{2} + \frac{2\,e^{2t}}{-1} + \frac{2\,e^{3t}}{2} = e^{t} - 2e^{2t} + e^{3t} \end{aligned}$$

Example 3. Find the inverse Laplace transform of,

$$(\mathcal{L}f)(s) = \frac{s}{s^2+a^2}$$

Solution: You could use the table, or alternatively use the method above. First we factor

$$\frac{s}{s^2+a^2}=\frac{s}{(s+\mathrm{i}a)(s-\mathrm{i}a)}$$

There are no repeated roots in the bottom and the top is a polynomial of degree less than the degree of the polynomial in the bottom, hence the method applies. Thus we can just write down the solution,

$$\begin{aligned} f(t) &= \frac{e^{st}\cdot s}{(s-\mathrm{i}a)}|_{s=-\mathrm{i}a}+\frac{e^{st}\cdot s}{(s+\mathrm{i}a)}|_{s=\mathrm{i}a} \\ &= \frac{e^{-\mathrm{i}at}\cdot(-\mathrm{i}a)}{-2\mathrm{i}a}+\frac{e^{\mathrm{i}at}\cdot\mathrm{i}a}{2\mathrm{i}a} \\ &= \frac{e^{\mathrm{i}at}+e^{-\mathrm{i}at}}{2}=\cos(a\,t) \end{aligned}$$

As an exercice try deriving $\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right)=\sin\ (a\ \ t)$ from the method above. Remember the formula

$$\sin(a\,x)=\frac{e^{\mathrm{i}ax}-e^{-\mathrm{i}ax}}{2\,\mathrm{i}}$$

for this.

2. Rational functions with multiple roots in the denominator.

This case is more difficult, and you can either use partial fraction or the method below.

Example 4. Find the Laplace inverse of

$$(\mathcal{L}f)(s)=\frac{1}{(s-1)(s-2)^2}$$

Solution: One method is to simply write down the partial fraction,

$$\frac{1}{(s-1)(s-2)^2}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

and then solve for A, B, C.

Another method is to reduce to the case of rational function with only simple roots in the denominator by using a trick. The trick is to introduce a new variable a, and express $1/(s-2)^2$ in terms of the derivative of $1/(s-a)$. Namely,

$$\frac{1}{(s-2)^2}=\frac{\mathrm{d}}{\mathrm{d}a}\cdot\frac{1}{s-a}|_{a=2}$$

because $(\mathrm{d}/\mathrm{d}a)(1/s-a)=1/(s-a)^2$ so when we estimate at $s=2$ we get $1/(s-2)^2$. With this trick at hand we write,

$$\frac{1}{(s-1)(s-2)^2}=\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{1}{s-1}\cdot\frac{1}{s-a}\right)|_{a=2}$$

Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left(\frac{1}{(s-1)(s-2)^2}\right) \\ &= \mathcal{L}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{1}{s-1}\cdot\frac{1}{s-a}\right)|_{a=2}\right) \end{aligned}$$

We are allowed to interchange the $\mathcal{L}^{-1}$ (Laplace inverse) with the differentiation operation, and so we keep on writting,

$$f(t) = \frac{\mathrm{d}}{\mathrm{d}a}\mathcal{L}^{-1}\left(\frac{1}{s-1}\cdot\frac{1}{s-a}\right)|_{a=2} \tag{1}$$

Now we know by the previous section that,

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s-1}\cdot\frac{1}{s-a}\right) &= \frac{e^{st}}{s-a}|_{s=1}+\frac{e^{st}}{s-1}|_{s=a} \\ &= \frac{e^t}{1-a}+\frac{e^{at}}{a-1} \end{aligned}$$

Therefore returning to (1) we have

$$\begin{aligned} f(t) &= \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{e^t}{1-a}+\frac{e^{at}}{a-1}\right)|_{a=2} \\ &= \frac{e^t}{(1-a)^2}+\frac{t\,e^{at}}{a-1}-\frac{e^{at}}{(a-1)^2}|_{a=2} \\ &= e^t+t\,e^{2t}-e^{2t} \end{aligned}$$

which is the desired solution.

Example 5. Find the inverse Laplace transform of

$$(\mathcal{L}f)(s)=\frac{s}{(s-1)^2(s-2)^2}$$

Solution: I will proceed more mechanically skipping more details as I've explained the method in the previous example. We have

$$\frac{s}{(s-1)^2(s-2)^2} = \frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\left(\frac{s}{(s-a)(s-b)}\right)|_{\substack{a=1\\b=2}}$$

Therefore,

$$f(t) = \mathcal{L}^{-1}\left(\frac{s}{(s-1)^2(s-2)^2} \right) = \frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathcal{L}^{-1}\left(\frac{s}{(s-a)(s-b)} \right)\Big|_{\substack{a=1\\b=2}}$$

Now,

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{(s-a)(s-b)} \right) &= \frac{s\,e^{st}}{(s-a)}\Big|_{s=b} + \frac{s\,e^{st}}{s-b}\Big|_{s=a} \\ &= \frac{b\,e^{bt}}{b-a} + \frac{a\,e^{at}}{a-b}\end{aligned}$$

Therefore returning to (2) we find,

$$\begin{aligned}f(t) &= \frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\left(\frac{b\,e^{bt}}{b-a} + \frac{a\,e^{at}}{a-b} \right)\Big|_{\substack{a=1\\b=2}} \\ &= \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{e^{bt} + b\,t\,e^{bt}}{b-a} - \frac{b\,e^{bt}}{(b-a)^2} + \frac{a\,e^{at}}{(a-b)^2} \right)\Big|_{\substack{a=1\\b=2}} \\ &= \left(\frac{e^{bt} + b\,t\,e^{bt}}{(b-a)^2} - 2\,\frac{b\,e^{bt}}{(b-a)^3} - 2\cdot\frac{e^{at} + t\,a\,e^{at}}{(a-b)^3} \right)\Big|_{\substack{a=1\\b=2}} \\ &= e^{2t} + 2\,t\,e^{2t} - 4\,e^{2t} + 2\,e^{t} + 2\,t\,e^{t}\end{aligned}$$

as you see in fact this is comparatively short. Comparatively to trying to do the same example via partial fractions,

$$\frac{s}{(s-1)^2(s-2)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

...